# Synchronization of Kuramoto Model Beyond Sinusoidal Interactions

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***For the Kuramoto model with the ReLU-Sin coupling, we provide the first sufficient initial conditions that lead to phase and frequency synchronizations, respectively.***

## The Kuramoto Model

For $1 \le i \le N$,

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^{N} \Gamma(\theta_j(t) - \theta_i(t)),$$

- $\omega_i$ : natural frequency.
- $\theta_i$ : phase
- $\dot{\theta}_i$ : frequency
- $\Gamma$ : a $2\pi$-periodic coupling function.

Describe the ***collective synchronization*** of fireflies, circadian rhythm, Josephson junctions, power grids, etc.

## Synchronization

***Phase synchronization***

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t) - 2k_{ij}\pi) = 0, \ \forall i, j.$$

***Frequency synchronization***

$$\lim_{t\to\infty} (\dot{\theta}_i(t) - \dot{\theta}_j(t)) = 0, \ \forall i, j.$$

## Main Results (Informal)

***"Competition leads to synchronization."***

Consider the ReLU-Sin coupling:

$$\Gamma(t) = k \max\{0, \sin(t)\}, \ k > 0.$$

Let $D(t) := \max_{i,j} |\theta_i(t) - \theta_j(t)|$.

1. ***Phase synchronization*** if $D(0) < \pi$ and oscillators are identical ($\omega_1 = \cdots = \omega_N$).
2. ***Frequency synchronization*** if $D(0) < \pi$ and $k \gg 0$. Moreover, $\dot{\theta}_i(t) \to \max_i \omega_i$.

## Challenge and Proof Ideas

***The challenge is the lack of oddness of the coupling function, making Lyapunov-based analyses fail.***

We ignore the max when $t \gg 0$ by the Order Lemma (informal):

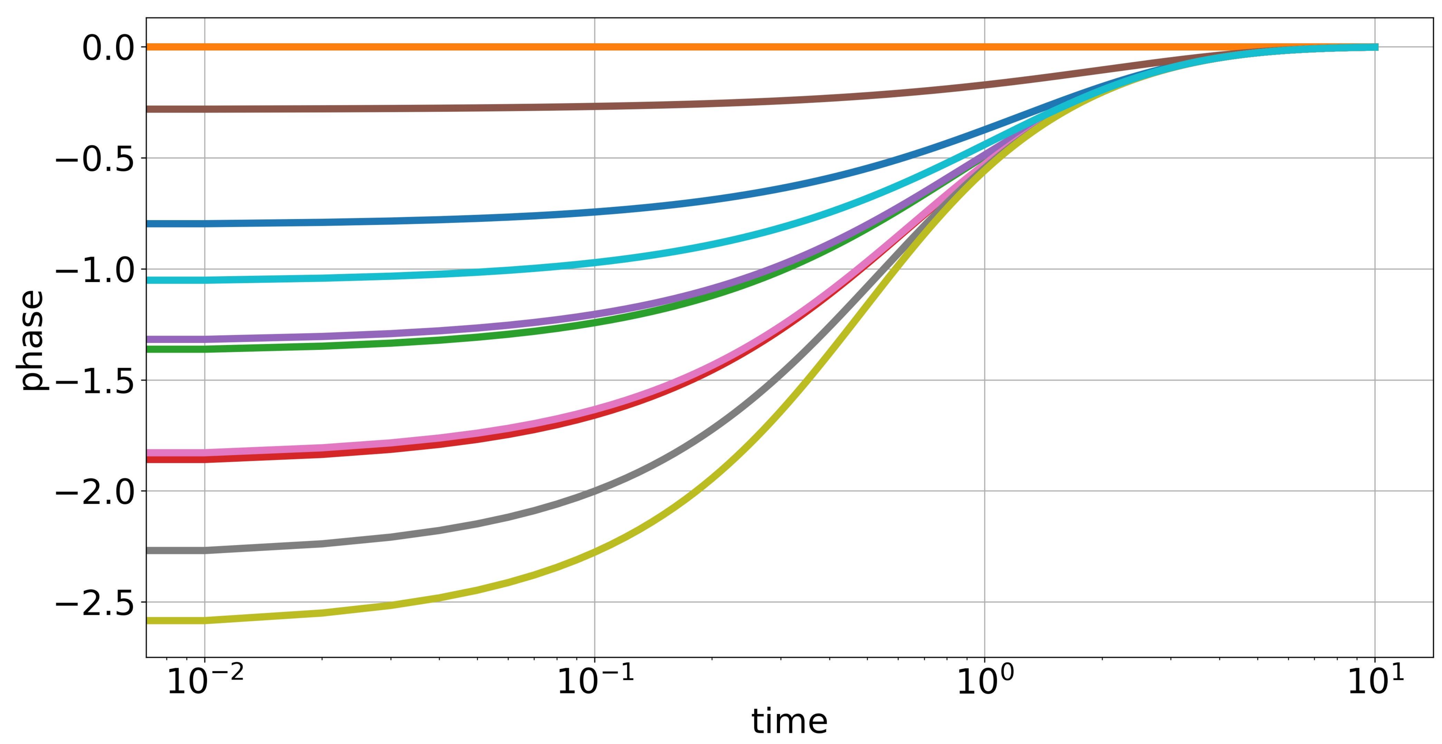
For any $\delta > 0$, if $D(0) < \pi$ and $k \gg 0$, then for $t \gg 0$, we have

1. $D(t) \le \delta$;
2. If $\omega_i > \omega_j$, then $\theta_i(t) \ge \theta_j(t)$.

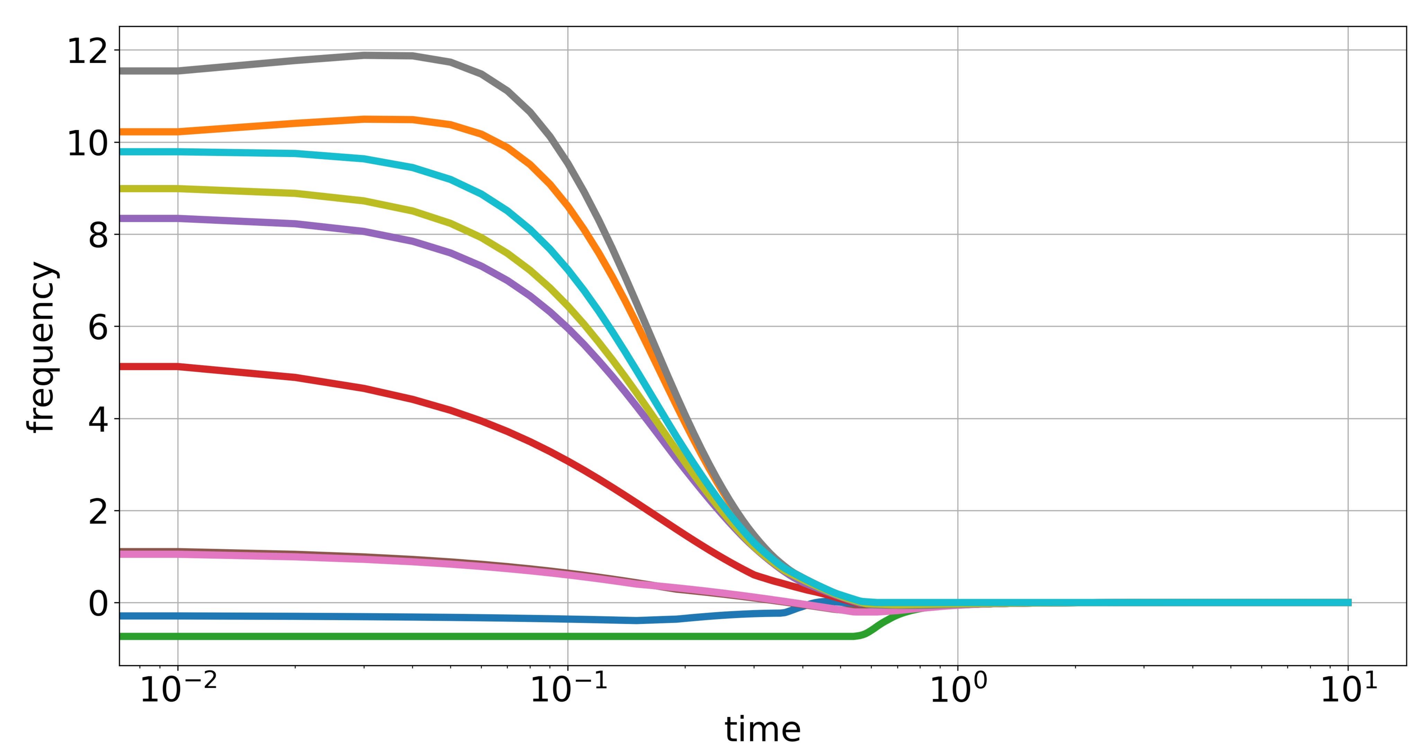
## Numerical Experiments

***Correctly validate our results.***

$N = 10, \omega_i = 0, D(0) = 0.265, k = 0.5$




$N = 10, \omega_i \in [-1, 0], D(0) = 0.255, k = 1.967$




## Other Results

Suppose $\Gamma$ is odd and analytic and the oscillators are identical. Then, they achieve ***frequency synchronization***.